Waves and Imaging Instruments
Lectures 9-14

3 Fourier Transforms

We now come to a mainly mathematical interlude, where we consider the decomposition of an arbitrary waveform into sine waves. This is known variously as Harmonic Analysis, Spectral Analysis or Fourier Analysis, and is the basis for the study of many problems in science. It is a powerful tool for analysing standing waves, interference, diffraction and imaging systems, which is why it is in this course, but you will undoubtedly find it useful in many other fields.

3.1 A word on notation

There are a lot of different ways of writing a sine wave of a given amplitude and phase, amongst them

$$A\cos(wt + f)$$
$$A\sin(wt + j + p/2)$$
$$B_1\cos(wt) + B_2\sin(wt)$$
$$C\exp(iwt) + C^*\exp(-iwt)$$

I will tend to mostly use the form

$$\Re(D\exp[iwt])$$

where $\Re$ denotes taking the real part, and D is a complex constant which encodes both amplitude and phase. Some authors just write $D\exp[iwt]$ and implicitly assume taking the real part. While this does make the formulae look simpler, I will try to avoid using this notation, because you can otherwise forget the implicit $\Re(\ )$.

3.2 Introduction: why analyse things in terms of sine waves?

Fourier Transforms are used in many places, from financial analysis, through MP3 recorders and image compression to radar systems, temperature diffusion, and many more fields. The question may arise as to why decomposing things into sine waves is applicable in so many different areas. The first part of the answer is "because we can"; i.e. it can be shown that *any* function (with a few unphysical exceptions) can be decomposed into a sum of sine waves. However, this is not a property unique to sine waves, for example it is also true that most functions can be decomposed into a sum of powers of x, i.e. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$ and there are many other possibilities including exotic functions like wavelets. It is a combination of properties of sine waves that makes Fourier decomposition particularly important in the analysis of many physical systems. These are:

[1] Sine waves are periodic. There are many systems that show periodicity, standing waves being the most obvious, the stock market being a less obvious one. Expressing a periodic function in terms of sine waves will allow us to capture the behaviour of an infinitely long time series in terms of a relatively small number of sine waves, i.e. it will be a compact representation which is easier to understand.

[2] Put a sine wave into a linear differential equation, and you will get a sine wave of the same frequency out. This is in fact a key property of sine waves. If we express a sine wave as $x = \Re\left(Ae^{iwt}\right)$, then the function $y = a\frac{dx}{dt} = \Re\left(aiwAe^{iwt}\right)$ is also a sine wave of the same frequency, just with a different complex constant in front, as is the function $y = a\frac{d^2x}{dt^2} = \Re\left(-aw^2Ae^{iwt}\right)$ or the function

$$y = ax + b\frac{dx}{dt} + c\frac{d^2x}{dt^2} + d\frac{d^3x}{dt^3}$$
$$=$$
$$=$$

and so on. This is a property unique to sine waves (and to their close cousins, the exponential functions Ae^{lt}).

The reason why this is a *useful* property is that very many systems can be analysed in terms of linear differential equations. The fact that sine waves "pass through" a linear differential system with only a change in the multiplicative constant in front of them is so important it has been given a name: sine waves are said to be the **eigenfunctions** of linear differential systems, and many results can be derived from this fact. One result is that we can use linear **systems theory** to simplify our understanding of a cascade of different systems. For example, if we are trying to analyse the effect of a car suspension system on the ride quality felt by a person inside the car, we need to consider the effect of the tyres, the suspension system, the transmission of vibrations through the car body and the springiness of the seats.

If we assume that all the parts of the car can be modelled as linear differential systems, then when we analyse the effect of a sinusoidal road surface, we find that the effect of the suspension is particularly easy to calculate:

If it is true (and it is) that *any* road surface can be considered as the sum of a set of sinusoidal road surfaces, then the recipe for understanding the effect of the system as a whole is:

(a) Split the road surface into a series of sine waves (Fourier Analysis)
(b) Calculate the effect of the system as a whole on each sine wave in turn (multiply by a constant, known as the **system gain** for that frequency or the **transfer function** at that frequency).
(c) Add the resulting sine waves back together (Fourier Synthesis).

The beauty of this method is that if a new system is added to stage (b), for example if the response of the human body to vibrations becomes an important factor, you just need to multiply the constant in step (b) by the appropriate constant for the human body and you have now solved the new problem. Also, if the type of road surface changes, you only need to change step (a) rather than redoing the whole calculation.

If we consider the specific problem of wave propagation, then the wave equation we have been dealing with so far turns out to be a special case: it is called a **dispersionless wave equation.** There exist other wave equations which are called dispersive wave equations. These are linear partial differential equations where sine waves of different frequencies propagate at different velocities. In this case, it is not easy to analyse the wave propagation in terms of the propagation of pulses, because the pulses tend to spread or disperse, hence the name. Instead, analysis in terms of sine waves becomes the only easy way to understand what is going on.

[3] Another property of sine waves that makes them useful is related to the property above, but is worth mentioning separately. It is that shifting or delaying a sine wave by a fixed amount in space or time is equivalent to multiplying it by a complex factor.

$$\Re\left\{A\exp[iw(t+\Delta t)]\right\} = \Re\left\{A\exp[iw\Delta t]\exp[iwt]\right\}$$

In many situations in wave propagation, many copies of a given wave that have been delayed by different amounts arrive at a given point. For example, consider the diffraction from a mask with 3 slits:

The resultant wave at that point is then, by superposition, the sum of these shifted copies. If we analyse this in terms of arbitrary pulses, then we

need to compute the effect of the shape of the pulse for each shift and this rapidly becomes quite complicated.
If we consider a sine wave, the shifted waves consist of the same exponential with different complex multiplying factors, so the final wave is just a scaled version of the original wave

$$\sum_i \Re\{A_i \exp[iw(t+\Delta t_i)]\} = \Re\left\{\exp[iwt]\sum_i A_i \exp[iw\Delta t_i]\right\}$$

This property will be seen to be important in diffraction calculations.

[4] One further property is that if the system we are analysing is *non-linear*, then analysing it in terms of sine waves is still useful. This is because a non-linear function of a sine wave tends to give a scaled version of the sine wave plus other sine waves at harmonic frequencies. For example, if we take a quadratic function acting on a sine wave:

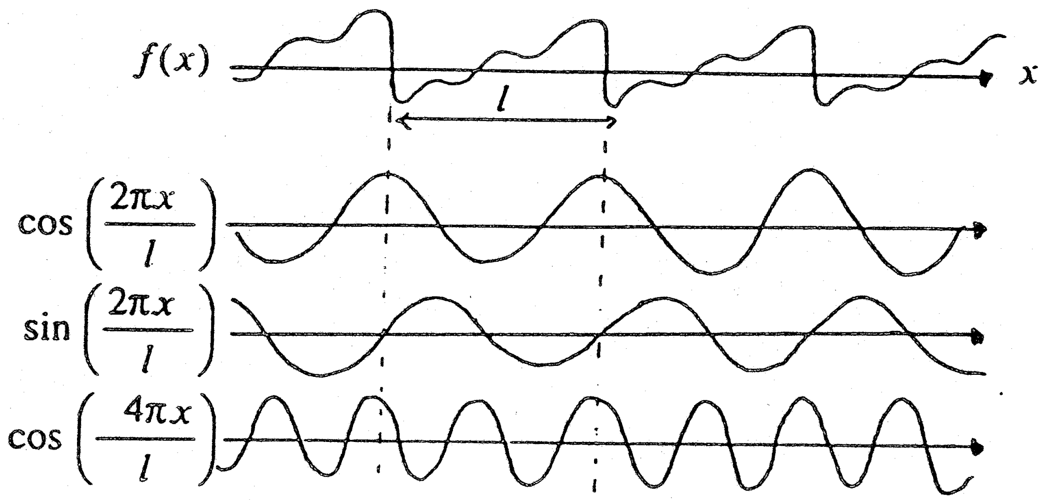
$$\begin{aligned}(\Re\{A\exp[iwt]\})^2 &= \left[\frac{1}{2}(A\exp[iwt]+A^*\exp[-iwt])\right]^2 \\ &= \frac{1}{4}(A^2\exp[2iwt]+(A^*)^2\exp[-2iwt]+2|A|^2) \\ &= \frac{1}{2}\left(|A|^2+\Re\{A^2\exp[2iwt]\}\right)\end{aligned}$$

3.3 Development of the argument

We will proceed as follows: first we will analyse periodic functions, because decomposition of these functions into sine waves would seem to be natural. The expansion is into a discrete set of sine waves called a Fourier Series. Then we consider functions that are not periodic, and show that they can be treated as the special case of a periodic function where the period of repetition is infinite. The expansion is then into a continuous set of sine waves, where the difference in frequency between one sine wave and its next nearest neighbour is infinitesimally small. This is called a Fourier Transform. We will then show there are many useful “tricks” that can be done with a Fourier Transform, chief amongst these being a convolution. Finally, we show that a Fourier series is in turn just a special case of a Fourier Transform!

3.3 Periodic functions: Fourier series

Consider a function $f(x)$ which is periodic in x, repeating itself in a distance l.



This function can, in general, be written as a sum of sinusoids:

$$f(x) = \frac{1}{2}A_o + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2pnx}{l}\right) + B_n \sin\left(\frac{2pnx}{l}\right)\right] \qquad (3.1)$$

The term corresponding to A_0 is the average value of the waveform. This term is some times referred to as the constant or D.C. term. The term corresponding to n=1 is commonly called the **fundamental** or **first harmonic** and all the terms corresponding to n>1 are called the **nth harmonic** or the **n-1th overtone**. The coefficients A_n and B_n are called the amplitudes of the harmonics. For the waveform shown, the average value is zero, so A_0=0, and the values for the harmonics are A1=1, B_1=1, A_2=1, B_2=0, and A_n, B_n=0 for n>2.

To find out what the values of the coefficients are for an arbitrary function, we make use of the fact that the integral over a complete cycle of products of sine waves with cos waves are zero:

$$\int_{-l/2}^{l/2} \cos\left(\frac{2pmx}{l}\right)\sin\left(\frac{2pnx}{l}\right)dx = 0 \qquad \text{all } m$$

and the products of cos waves with cos waves and sine waves with sine waves integrate to zero except when they are of the same frequency

$$\int_{-l/2}^{l/2} \cos\left(\frac{2pmx}{l}\right)\cos\left(\frac{2pnx}{l}\right)\mathrm{d}x = 0 \qquad m \neq n$$
$$= l/2 \qquad m = n$$

$$\int_{-l/2}^{l/2} \sin\left(\frac{2pmx}{l}\right)\sin\left(\frac{2pnx}{l}\right)\mathrm{d}x = 0 \qquad m \neq n$$
$$= l/2 \qquad m = n$$

Multiplying equation (3.1) through by $\cos\left(\frac{2pmx}{l}\right)$ and integrating, we get

$$\int_{-l/2}^{l/2} f(x)\cos\left(\frac{2pmx}{l}\right)dx = \int_{-l/2}^{l/2} \frac{1}{2}A_o\cos\left(\frac{2pmx}{l}\right)dx$$
$$+\sum_{n=1}^{\infty}\left[\begin{array}{l}\int_{-l/2}^{l/2} A_n\cos\left(\frac{2pnx}{l}\right)\cos\left(\frac{2pmx}{l}\right)dx \\ +\int_{-l/2}^{l/2} B_n\sin\left(\frac{2pnx}{l}\right)\cos\left(\frac{2pmx}{l}\right)dx\end{array}\right]$$

Most of the terms integrate to zero giving

$$A_n = \frac{2}{l}\int_{-l/2}^{l/2} f(x)\cos\left(\frac{2pnx}{l}\right)\mathrm{d}x \qquad n = 0,1,2... \qquad (3.2)$$

Similarly, multiplying by $\sin\left(\frac{2pmx}{l}\right)$ and integrating gives

$$B_n = \frac{2}{l}\int_{-l/2}^{l/2} f(x)\sin\left(\frac{2pnx}{l}\right)\mathrm{d}x \qquad n = 1,2... \qquad (3.3)$$

An alternative way of presenting a Fourier series is in terms of complex exponentials. This has the advantage of being more symmetric and therefore easier to remember. Computations with complex exponentials tend to be easier than using cos's and sine's. Remembering that

$$\cos q = \left(e^{iq} + e^{-iq}\right)/2 \text{ and } \sin q = \left(e^{iq} - e^{-iq}\right)/2\mathrm{i}$$

and

$$\int_{-l/2}^{l/2} e^{-i2pmx/l} e^{i2pnx/l} \mathrm{d}x = 0 \qquad m \neq n$$

$$= l \qquad m = n$$

we can write

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2pnx/l}$$

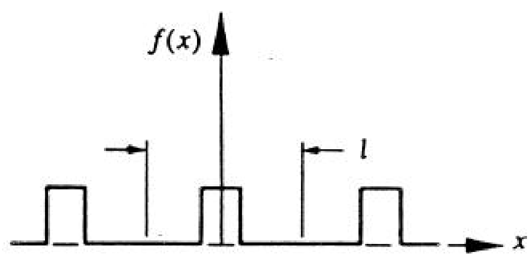
where

$$C_n = \frac{1}{l} \int_{-l/2}^{l/2} f(x) e^{-i2pnx/l} \mathrm{d}x$$

We can identify $(2pn/l)$ as the **wave-number** k of the harmonic wave component, so

$$C(k) = \frac{1}{l} \int_{-l/2}^{l/2} f(x) e^{-ikx} \mathrm{d}x$$

Example: The function shown below repeats in distance *l*. Find the complex Fourier Series coefficients for this function



Answer:

In the range $-l/2 < x < l/2$ we have

$$f(x) = A \qquad -l/8 < x < l/8$$

$$= 0 \qquad l/8 < |x| < l/2$$

Then

$$
\begin{aligned}
C(k) &= \frac{1}{l}\int_{-l/8}^{l/8} A e^{-ikx}dx \\
&= \frac{A}{l}\left[\frac{e^{-ikx}}{-ik}\right]_{-l/8}^{l/8} \\
&= \frac{A}{ikl}\left(e^{ikl/8} - e^{-ikl/8}\right) \\
&= \frac{2A}{kl}\sin\left(kl/8\right)
\end{aligned}
$$

Defining the **sinc function** as

$$\mathrm{sinc}(x) = \frac{\sin x}{x} \tag{3.4}$$

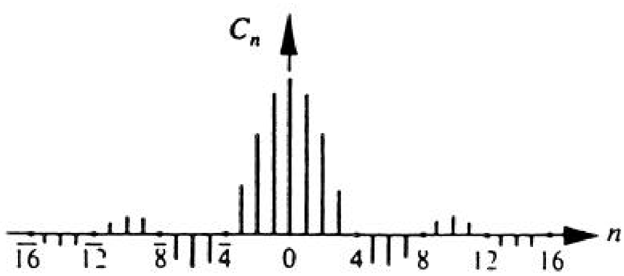
we get

$$C(k) = \frac{A}{4}\mathrm{sinc}(kl/8)$$

Alternatively we can write in terms of the nth harmonic coefficient

$$C_n = \frac{A}{4}\mathrm{sinc}\left(2pn/8\right) = \frac{A}{4}\mathrm{sinc}\left(np/4\right) \tag{3.5}$$

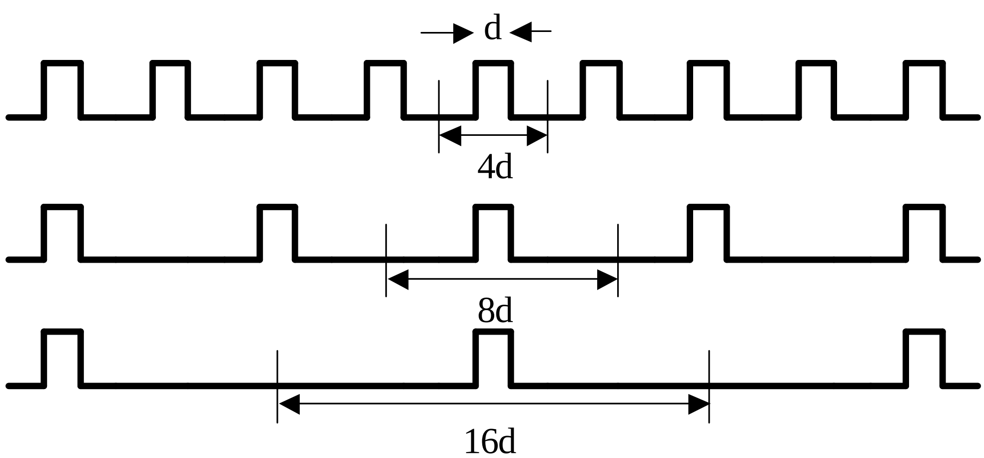
This is zero whenever *n* is an integral multiple of 4. We can plot the coefficients as a function of coefficient number as



3.5 From Fourier Series to Fourier Transforms

To reiterate the results from the last section, we can represent a periodic function of x as the sum of a series of sinusoids. We can plot the coefficients C_n of the sinusoids as a function of the coefficient number n, or of the wave-number k, where $k = 2pn/l$ and l is the repeat interval. We can consider these as two representations of the same function, one in the x-domain, and one in the k-domain. The x-domain function is continuous, whereas the k-domain function only has values at a discrete set of points.

We now come to generalise this result to functions that are not repetitive. We do this by defining a function which does not repeat as the limit of a repetitive function as the repeat interval tends towards infinity. For simplicity we consider the Fourier representation of a fixed-width square pulse as the interval between pulses gets longer and longer:



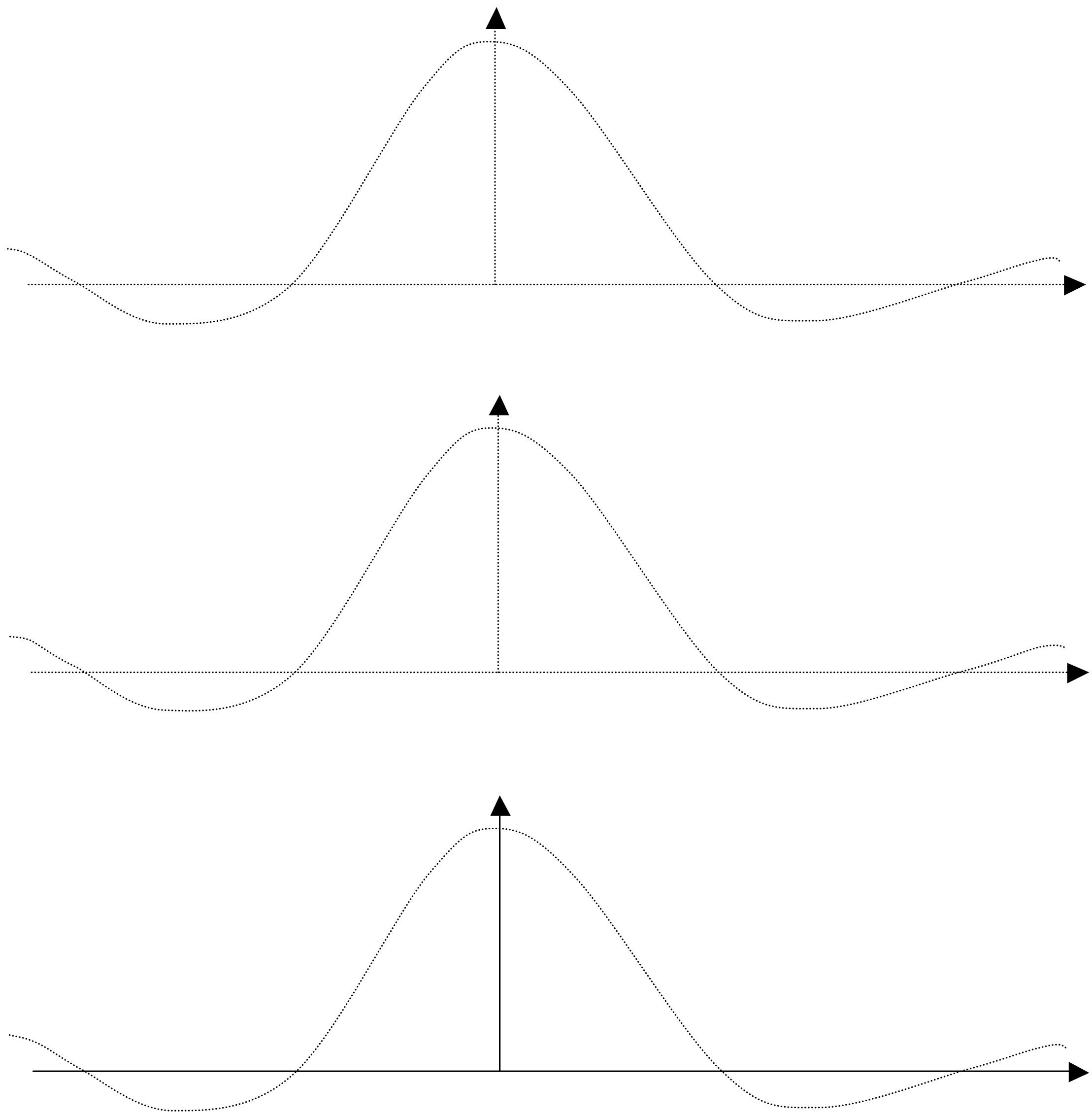
The n'th Fourier coefficient for a repeat interval of l will be given by

$$
\begin{aligned}
C_n &= \frac{1}{l}\int_{-d/2}^{d/2} e^{-2pinx/l}dx \\
&= \left[e^{-2pinx/l}/2pni\right]_{-d/2}^{d/2} \\
&= \sin(pnd/l)/pn \\
&= \frac{d}{l}\mathrm{sinc}(pnd/l)
\end{aligned}
$$

or as a function of k as

$$C(k) = (d/l)\mathrm{sinc}(\mathrm{kd/2})$$

We can plot these coefficients as a function of wave-number k for different values of the repeat interval l:



As the repeat interval gets longer and longer, the spacing between the Fourier coefficients gets smaller and smaller. In the limit that the repeat interval is infinite, the k-space representation goes from being a set of coefficients at a discrete set of frequencies to being a continuous function of k. We call this continuous k-space function F(k) the **Fourier Transform** of the x-space function f(x). We normalise F(k) so that

$$\begin{aligned} F(k) &= \lim_{l \to \infty} C(k) l \\ &= \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx \end{aligned}$$

For the square pulse in the example above, we get

$$F(k) = d\,\mathrm{sinc}(\mathrm{kd/2})$$

To get back to the x-space function from the k-space "coefficients", our summation becomes an integration:

$$f(x) = \lim_{l \to \infty} \sum_{n=-\infty}^{\infty} C_n l \exp(ikx) = \frac{1}{2p} \int_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

where the factor of $1/2p$ comes from the fact that the k-space density of frequency points is given by

$$\begin{aligned} k &= 2pn/l \\ \therefore dk &= 2pdn/l \end{aligned}$$

Thus we have two almost-symmetric equations defining a **forward Fourier Transform**

$$F(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx$$

and the **inverse Fourier Transform**

$$f(x) = \frac{1}{2p} \int_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

We can make these equations more symmetric by writing them in terms of the **spatial frequency**

$$s = k/2p$$

In this case we get

$$F(s) = \int_{-\infty}^{\infty} f(x) \exp(-2pisx) dx \qquad (3.6)$$

$$f(x) = \int_{-\infty}^{\infty} F(s) \exp(2pisx) ds \qquad (3.7)$$

Unfortunately, there is no one standardised way of normalising the Fourier Transform, and there are two other commonly-used pairs of equations:

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx$$
$$f(x) = \int_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

and

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

To avoid confusion, we will use equations (3.6) and (3.7) whenever we do Fourier Transforms.

If x is a spatial coordinate with units of metres then k (or s) has units of 1/distance (m^{-1}) and so k-space is often called **reciprocal space** (x-space is called **real space).** If x is replaced by a time coordinate in units of seconds then s can be replaced by a frequency f (or k with an angular frequency $\omega = 2\pi f$) with units 1/time (Hz). The two representations are then called the **time-domain** representation and the **frequency-domain** representation of a given waveform respectively.

We will adopt the convention that lower case letters denote real-space or time-domain functions and upper case letters denote the Fourier Transforms of these functions, e.g.

$$F.T.\{g(x)\} = G(s)$$
$$I.F.T.\{H(f)\} = h(t)$$

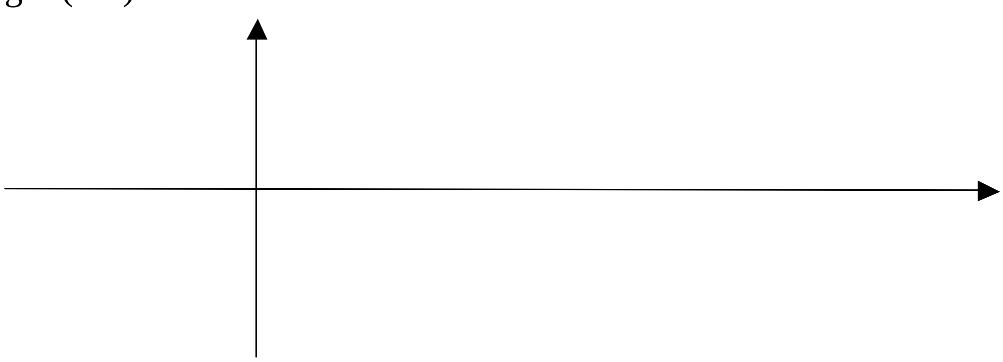
An alternative convention is to use a *tilde* to represent the transformed function $\tilde{g}(s)$. We will use both conventions as necessary.

3.6 Interpretation of the Fourier transform of a function

If g(t) is a time-domain waveform, then G(f) for some particular frequency f_1 can be interpreted in terms of the output of a narrow-band filter tuned to frequency f_1:

$G(f_1)$ will in general be a complex number which we can represent as a vector in the Argand diagram.

The magnitude of this vector $|G(f_1)|$ will give the amplitude of the sine wave output of the filter, whereas the phase of the vector $q = \arg[G(f_1)]$ tells us the phase shift of the sine wave with respect to one centred at the origin (t=0).



ASIDE: to be strictly accurate, we need to represent any real (in both senses of the word!) filter as a combination of two narrow functions of frequency, one at frequency f_1 and one at the symmetric frequency $-f_1$. If g(t) is a real function, then we can show that G(f) is **Hermitian**, i.e. that $G(-f_1) = G(f_1)^*$ for all f_1. Hence the output of the filter will be proportional to

$$G(f_1)\exp(2pif_1t) + G(f_1)^* \exp(-2pif_1t) = \Re\{2G(f_1)\exp(2pif_1t)\}$$

This function is a sine wave with a maximum when $2pif_1t = -q$
END OF ASIDE

Example: A capacitor is charged very rapidly and then discharged through a resistor. The voltage across the capacitor is fed to a bank of narrowband electronic filters tuned to a range of different frequencies. Sketch the amplitude and phase of the filter outputs as a function of the frequency of the filter.

Answer: The voltage across a capacitor of capacitance C discharging through a resistance R is $v(t) = v_0 \exp(-t/t)$ where v_0 is the initial voltage and the time constant t is given by $t = RC$.

The frequency-domain representation of this function is

$$V(f) = \int_{-\infty}^{\infty} v(t)\exp(-2pift)dt$$

$$= \int_{0}^{\infty} v_0 \exp(-t/t)\exp(-2pift)dt$$

$$= \int_{0}^{\infty} v_0 \exp(-[1/t + 2pif]t)dt$$

Writing $1/t = 2pf_0 = w_0$ we get

$$V(f) = \left[-\frac{v_0 \exp(-2p[f_0 + if]t)}{2p[f_0 + if]} \right]_0^{\infty}$$

$$= \frac{v_0}{2p[f_0 + if]} = \frac{v_0}{w_0 + iw}$$

We notice that this is the same functional form as the frequency response of a low-pass filter, and one way to make low-pass filters is using a resistor-capacitor network! The magnitude of this function $|V(f)|$ is approximately constant at v_0/w_0 at low frequencies $w << w_0$ and approximately proportional to $1/w$ at high frequencies $w >> w_0$.

This represents the amplitude of the output of a filter as a function of the filter frequency.

The phase of this function, and hence of the output of a filter at a given frequency, is approximately zero at low frequencies, and 90° at high frequencies:

3.7 Delta functions

The narrowband filters in the previous example are an instance of a common requirement, namely to take a small sample of a continuous function, whether taking a narrow range of frequencies around some centre frequency or taking a sample of the instantaneous value of a continuous time series. We can represent this as taking an integral over a very small region of the function being sampled. We can invent a sampling function which multiplies the input function before the integration takes place:

The best sort of sampling function would be one that is as narrow as possible, but still contains finite area. We can imagine this as the limit of a "tophat" function of unit area as the width of the tophat gets smaller and smaller.

The limit of this sequence, a "spike" that is infinitely narrow and infinitely high, we call a **delta function**. We cannot write down a functional form for this function, because of the infinities involved, but we can define it in terms of its integral properties: a delta function $d(x)$ is that function which, when multiplied by any other function and integrated, "picks out" the value of the other function at $x = 0$

$$\int_{-\infty}^{\infty} g(x) d(x) dx = g(0) \tag{3.8}$$

We can also use a shifted delta-function to pick out the value of a function at a non-zero value of x:

$$\int_{-\infty}^{\infty} g(x) d(x - x_1) dx = g(x_1) \tag{3.9}$$

The inverse Fourier Transform of a delta-function in frequency space is a complex exponential in the time domain, because the delta-function picks out a single frequency out of the infinite set of possible frequencies:

$$\text{I.F.T}\{d(f-f_0)\}=\int_{-\infty}^{\infty}\exp(2pift)d(f-f_0)df=\exp(2pif_0t) \qquad (3.9)$$

Hence the Fourier Transform of a complex exponential is a delta function.

3.8 Convolution

In our last example, we considered a capacitor that was charged very rapidly and then allowed to discharge through a resistor. We can think of the charging process as a being very sharp pulse of current, injecting a charge of $I\Delta t=Q=v_0C$, where v_0 is the voltage on the capacitor at the beginning of the discharge phase. It is natural therefore to represent this sharp pulse of current as a delta-function in time, since it has finite area (the charge) but lasts for no time (or at least a time which is so short that we cannot tell, or do not care, how long it lasts). There are many similar so-called **impulsive** phenomena, for example a mechanical impulse (like a sharp kick) that imparts a finite momentum in an arbitrarily short time. A system's response to an impulse typically lasts for a finite time, for example in the case of the capacitor-resistor network we get an exponential decaying voltage with a characteristic time constant $t = RC$. We call this the **impulse response function**, r(t).

If we now consider what happens if we give the system two kicks in succession, we will get the sum of two copies of the impulse response function shifted in time with respect to each other.

We can imagine the system's response to a general input waveform i(t) as being the response to a stream of impulses spaced by a time short compared to the characteristic timescale of the impulse response function.

The output o(t) at any given time t is then the sum of many samples of the response function shifted by many different time intervals. Each sample is scaled by the size of the impulse generating it, and hence by the value of the input function at a given time in the past.

The output is therefore given by

$$\begin{aligned} o(t) &= i(t)r(0) + i(t-\Delta t)r(\Delta t) + i(2-\Delta t)r(2\Delta t) + \ldots \\ &= \sum_{n=-\infty}^{\infty} i(t-n\Delta t)r(n\Delta t) \end{aligned}$$

We have taken the liberty of changing the lower limit of n from 0 to $-\infty$, because r(t) is zero for t<0, and we are anticipating applying our results to spatial impulse response functions, where there can be a response in both the x>0 and x<0 directions (in temporal response functions, this would correspond to clairvoyance).

Taking the limit as Δt tends to zero, we get

$$o(t) = \int_{-\infty}^{\infty} i(t-t')r(t')dt \tag{3.11}$$

The operation in equation (3.11) is given a name: **convolution.** The function i(t) is said to be **convolved** with the function r(t) and it is written

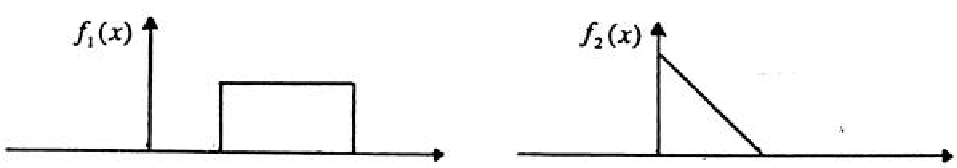
$$\begin{aligned} o(t) &= i(t) * r(t) \\ &= r(t) * i(t) \end{aligned}$$

Equations of this form are extremely common in so-called **linear shift-invariant systems.**

- **Shift invariant** system Þ "kick" the system at widely-separated times (or spaces) and you get the same response repeated.
- **Linear** shift invariant system Þ kick the system at closely-spaced times and you get the **sum** of the responses.

Convolution is important in imaging systems where convolution can be thought of as a blurring process. The impulse response function is called the **point spread function (PSF)**. The most dramatic illustration of the effects of changing the PSF was the initial optical error with the Hubble Space Telescope and its subsequent repair.

To get a visual idea of the meaning of a convolution, let us take as an example the convolution of two functions $f_1(x)$ and $f_2(x)$, where f_1 is a tophat function and f_2 is a "wedge" (a simplified version of the exponential function in our previous example)

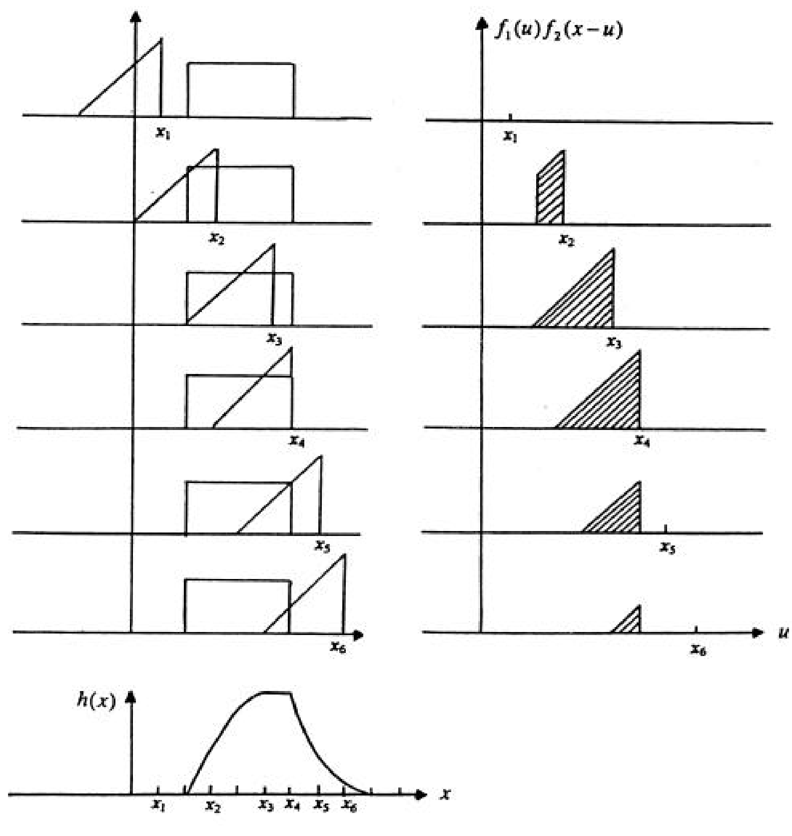


The convolution to be evaluated is

$$h(x) = \int_{-\infty}^{\infty} f_1(u) f_2(x-u) du$$

The graphical procedure for doing this convolution is

1. Flip one of the functions about the y axis.
2. Slide this function past the other function in a series of steps.
3. At each step (corresponding to one value of x), multiply the two functions together.
4. Integrate the area under the resulting curve (the u integral). This *roughly* corresponds to the overlap area of the two curves.
5. Plot the resulting value as the y-value of the convolution for the given value of x.
6. Repeat for a new value of x



Example: Calculate the convolution of a "tophat" function with itself.

Solution:

When one of the functions is considerably narrower than the other, we can think of a convolution as a smoothing or averaging process: the value of the convolution at a given point is a running average of the neighbouring points in the wider of the two functions, with the weights depending on the shape of the narrower function. Hence convolution can be thought of as a filtering process.

Convolving a function with a delta function at the origin has no effect, since the delta function is clearly the perfect impulse response function. However convolving a function with a shifted delta function will shift that function by the amount the delta function has been shifted:

$$\begin{aligned} f(x) * d(x - x_1) &= \int_{-\infty}^{\infty} f(u) d(u - [x - x_1]) du \\ &= f(x - x_1) \end{aligned} \qquad (3.12)$$

3.9 Convolution and Fourier Transforms

The reason why we have mentioned convolution in a section on Fourier Transforms is that the convolution is particularly easy to do in with the Fourier Transform: the Fourier Transform of the convolution of two functions is obtained by simply multiplying their Fourier Transforms:

$$F.T.\{f(x) * g(x)\} = F.T.\{f(x)\} \times F.T.\{g(x)\} \qquad (3.13)$$

i.e. if $h(x) = f(x) * g(x)$, then $H(s) = F(s)G(s)$. This is such an important result it is worth proving. This is easiest to do by calculating the inverse Fourier Transform of F(s)G(s):

$$\begin{aligned} h(x) &= \int_{-\infty}^{\infty} F(s)G(s)\exp(2pisx)ds \\ &= \int_{-\infty}^{\infty} \begin{matrix} \int_{-\infty}^{\infty} f(x_1)\exp(-2pisx_1)dx_1 \int_{-\infty}^{\infty} g(x_2)\exp(-2pisx_2)dx_2 \\ \exp(2pisx)ds \end{matrix} \\ &= \int \underset{-\infty}{\overset{\infty}{\iint}} f(x_1)g(x_2)\exp(-2pis[x_1 + x_2])\exp(2pisx)dx_1dx_2ds \end{aligned}$$

We can do the integral over s by making use of the fact that the Fourier Transform of a complex exponential is a delta function:

$$\begin{aligned} h(x) &= \underset{-\infty}{\overset{\infty}{\iint}} f(x_1)g(x_2)d(x - x_1 - x_2)dx_1dx_2 \\ &= \int_{-\infty}^{\infty} f(x_1)g(x - x_1)dx_1 = f(x) * g(x) \end{aligned}$$

This result (called the **convolution theorem**) means that any system that obeys a convolution relationship between its inputs and outputs can be easily analysed in the Fourier domain. Any sine waves at the input are transformed to sine waves at the output simply by multiplying by the Fourier Transform of the impulse response function:

$$\begin{aligned} o(t) &= i(t) * r(t) \\ O(f) &= I(f)R(f) \end{aligned}$$

The Fourier Transform R(f) of the impulse response function r(t) is called the **frequency response function** or the **modulation transfer function** of the system. The value of R(f) at a given frequency tells us how much a sine wave at that frequency will be amplified or attenuated when passing through the system. When several such systems are cascaded together we merely need to multiply the frequency response functions of the systems together, i.e. we convolve their impulse response functions.

An example is the R-C circuit analysed earlier. We showed that this has an impulse response that is a decaying exponential. The frequency response function of this circuit is the Fourier Transform of this exponential and we have already shown that this is the functional form of a low-pass filter.

If we cascade two such filters, we get a filter that "cuts off" twice as fast:

3.10 Doing Fourier Transforms in practice

For the majority of this course, we will try to avoid wherever possible having to do Fourier Transforms by doing integrals. Instead, we will make use of a few "building-block" Fourier Transforms and combine them using the various properties of the Fourier Transform that we will elucidate in this section.

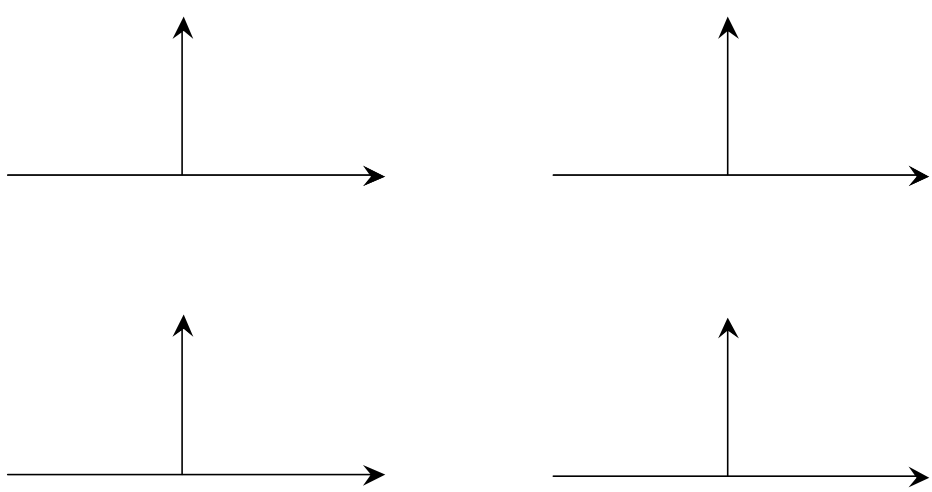
First a table of building-block transforms, stated in their simplest possible forms:

f(x)	F(s)
Shifted delta-function $d(x - x_1)$	**Complex exponential** $\exp(-2pix_1s)$
Tophat function $f(x) = 1$ where $-\frac{1}{2} < x < \frac{1}{2}$ $= 0$ otherwise	**Sinc function** $\sin(ps)/(ps)$
Exponential function $f(x) = \exp(-x)$ where $x > 0$ $= 0$ elsewhere	**Low-pass filter** $\frac{1}{1+2pis}$

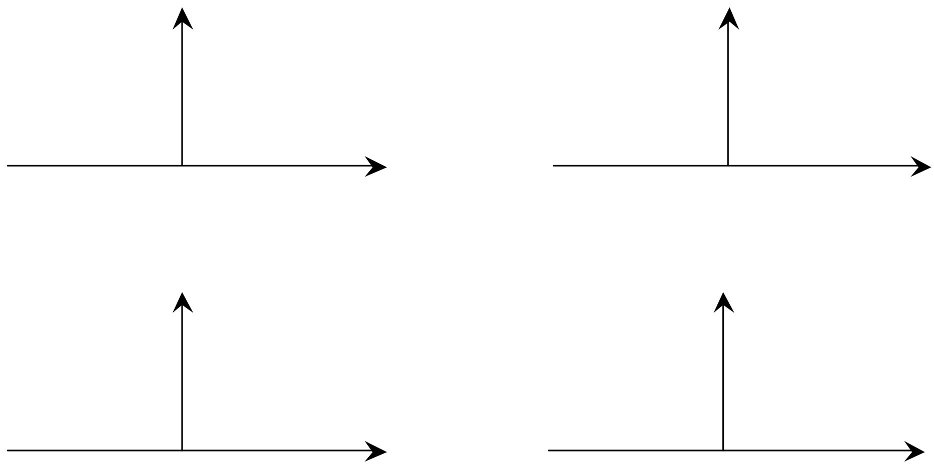
Gaussian $\exp(-x^2)$	**Gaussian** $\sqrt{p}\,\exp(-p^2 s^2)$
Comb function $\sum_{n=-\infty}^{\infty} d(x-n)$	**Comb function** $\sum_{n=-\infty}^{\infty} d(x-n)$

Properties of the Fourier Transform:

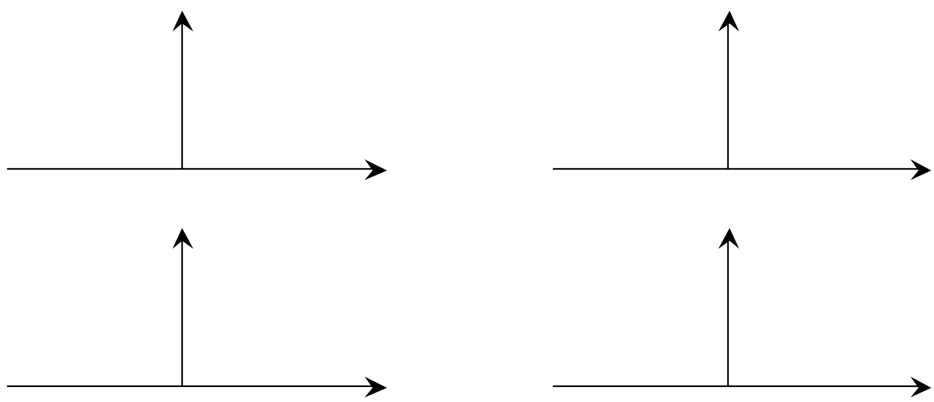
1. Reciprocity: If you know how to do the forward transform of a function, you can do the inverse transform of the same function, with the addition of a minus sign. If the F.T. of f(x) is g(s), then the I.F.T. of f(s) is g(-x).



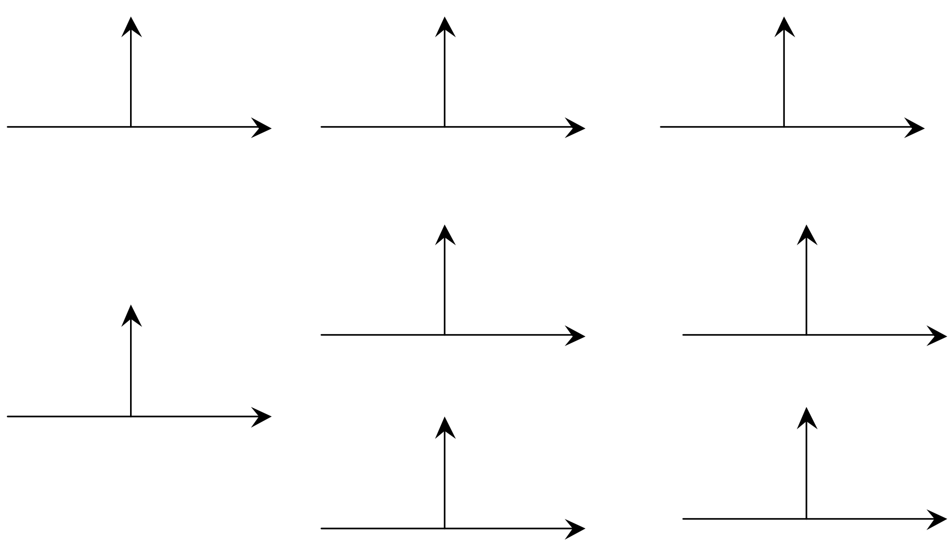
2. Linearity: $F.T\{af(x)+bg(x)\}=aF(s)+bG(s)$



3. Scaling law: $F.T.\{f(x/a)\}=aF(as)$



4. The convolution theorem: $F.T.\{f(x)*g(x)\}=F(s)G(s)$ and its inverse $F.T.\{f(x)g(x)\}=F(s)*G(s)$



Example 1: What is the Fourier Transform of (a) $\cos(2pf_0t)$ (b) $\sin(2pf_0t)$?

Solution: (a) We know that

$$\text{F.T.}\left\{\exp(2pif_0t)\right\} = d\,(f - f_0)$$

and that

$$\cos(2pf_0t) = \tfrac{1}{2}[\exp(2pif_0t) + \exp(-2pif_0t)],$$

so from linearity,

$$\text{F.T.}\left\{\cos(2p\mathrm{f}_0t)\right\} = \tfrac{1}{2}[d\,(f - f_0) + d\,(f + f_0)]$$

(b) Similarly, the F.T. of $\sin(2pf_0t)$ is

Example 2: What is the Fourier Transform of $1 - \exp(-x^2)$?

Solution: From linearity we know that we just need to subtract the F.T. of the Gaussian from the F.T. of 1. But what is the F.T. of 1? The answer is to notice that the complex exponential $\exp(2pis_0x)$ is unity everywhere when $s_0=0$. So the F.T. of 1 is $d(s)$, and the F.T. of $1 - \exp(-x^2)$ is $d(s) - \sqrt{p}\exp(-p^2s^2)$

Example 3: In **amplitude modulated** (A.M.) radio transmission, the transmission consists of an electromagnetic carrier wave $E(t) = A\cos(2pf_0t)$, where f_0 is a radio frequency of several megahertz, multiplied by the signal to be transmitted. If the signal to be transmitted is a Gaussian pulse of FWHM (Full Width to Half Maximum) t, what is the frequency representation of the radio wave?

Solution: The signal is a classic **wave packet,** $E(t) = \exp(-[t/a]^2) \times A\cos(2pf_0t)$, where a is related to the width of the Gaussian.

From the convolution theorem, we know that the Fourier Transform of this function will be the convolution of the F.T. of the Gaussian and the F.T. of the cos function. From the scaling property of the F.T., we get that the F.T. of the Gaussian is $a\exp(-[apf]^2)$, and from the previous example, we know that the F.T. of the cos wave is 2 delta-functions. Convolving with a delta function is easy, since we just shift the function being convolved by the shift of the delta function:

We have been given the FWHM, which is defined as the distance between the points at which the Gaussian has fallen to half of its peak value.

So we have $\exp(-[t\ /2a]^2) = \frac{1}{2}$ i.e. $t\ /2a = \ln 2$, hence $a = t\ /(2\ln 2)$.
The Gaussian in frequency space is proportional to $\exp(-[f\ /b]^2)$ where $b = 1/pa$ so the FWHM of this Gaussian is $2b\ln 2 = 2\ln 2/pa = 1/pt$.

This example illustrates the general result that the frequency spread of a function is inversely proportional to the time spread of a function. If the radio station wants to transmit very fast pulses, it must get a large frequency allocation. If these pulses denote digital ones and zeros, then the rate at which information can be transmitted will be proportional to the **bandwidth** available – this is one of the fundamental results of **information theory**.

Another of the consequences of this inverse proportionality is the quantum uncertainty principle. The quantum representation of the position of a particle is as wavepacket in space, and the momentum of the particle is represented as the Fourier Transform of this wavepacket. If the position of the particle is confined to a small region in space, then the momentum spread is large, and vice-versa.

3.11 Fourier transforms and symmetry

The Fourier transform can be written in terms of cos and sin as:

$$F(s) = \int_{-\infty}^{\infty} f(x)\left[\cos(2psx) + i\sin(2psx)\right]dx \qquad (3.13)$$

The different symmetries of the cosine and sine can be used to derive a set of **symmetry rules** for Fourier transforms. These symmetries allow us to infer the general properties of a system involving a Fourier transform without having to explicitly evaluate any integrals, and can be used the check the results of a transform calculation.

We say that a function has **even** symmetry if $f(x) = f(-x)$ and it has **odd** symmetry if $f(x) = -f(-x)$, or alternatively we can call them **symmetric** and **antisymmetric** respectively.

The cosine is an even function, and the sine function is an odd function. Thus we get the following results:

$$\int_{-\infty}^{\infty}\cos(2psx)E(x)dx \neq 0$$

$$\int_{-\infty}^{\infty}\cos(2psx)O(x)dx = 0$$

$$\int_{-\infty}^{\infty}\sin(2psx)E(x)dx = 0$$

$$\int_{-\infty}^{\infty}\sin(2psx)O(x)dx \neq 0$$

We can substitute these results into equation (3.13) to show that the Fourier transform of any function which is real and even is itself real and even, and that the Fourier transform of a function which is real and odd is imaginary and odd.

Now <u>any</u> real function $f(x)$ can be written as the superposition of a real, even function and a real, odd function:

$$f(x) = \frac{1}{2}\big(f(x) + f(-x)\big) + \frac{1}{2}\big(f(x) - f(-x)\big)$$

This means that the F.T. of any real function is the sum of an even real part and an odd imaginary part, i.e. it is **Hermitian** $f(x) = f^*(-x)$.

The relationships between the symmetries of a function and its (forward or inverse) Fourier Transform can be summarised as:

$$\begin{aligned} \text{real, even} &\Leftrightarrow \text{real, even} \\ \text{real, odd} &\Leftrightarrow \text{imaginary, odd} \\ \text{real} &\Leftrightarrow \text{Hermitian} \end{aligned}$$

<u>3.12 Higher-dimensional Fourier Transforms</u>

A Fourier transform can be defined in any number dimensions. In two dimensions the forward transform is defined as

$$F(s_x, s_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\exp(2pi[s_x x + s_y y])dxdy \qquad (3.14)$$

and the inverse transform is defined as

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(s_x, s_y)\exp(2pi[s_x x + s_y y])ds_x ds_y \qquad (3.15)$$

Noticing that $s_x x + s_y y$ can be written as $\vec{s}.\vec{x}$, where $\vec{s} = (s_x, s_y)$ and $\vec{x} = (x, y)$, we can also write these equations in vector form as

$$\begin{aligned} F(\vec{s}) &= \int_{\text{all space}} f(\vec{x})\exp(2pi\vec{s}.\vec{x})dA \\ f(\vec{x}) &= \int_{\text{all space}} F(\vec{s})\exp(2pi\vec{s}.\vec{x})dA_s \end{aligned}$$

where dA is an area element in $\vec{x}$ space and dA_s is an area element in $\vec{s}$ space.

Any one spatial-frequency component, $\exp(2pi\vec{s}_0.\vec{x})$ is a 2-dimensional sine wave in x space with the wave crests perpendicular to $\vec{s}_0$ and crest separation $1/|\vec{s}_0|$.

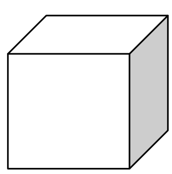
If $f(x,y)$ is **separable**, i.e. if

$$f(x,y) = f_x(x) f_y(y)$$

then we can do the transform as the product of 2 one-dimensional transforms:

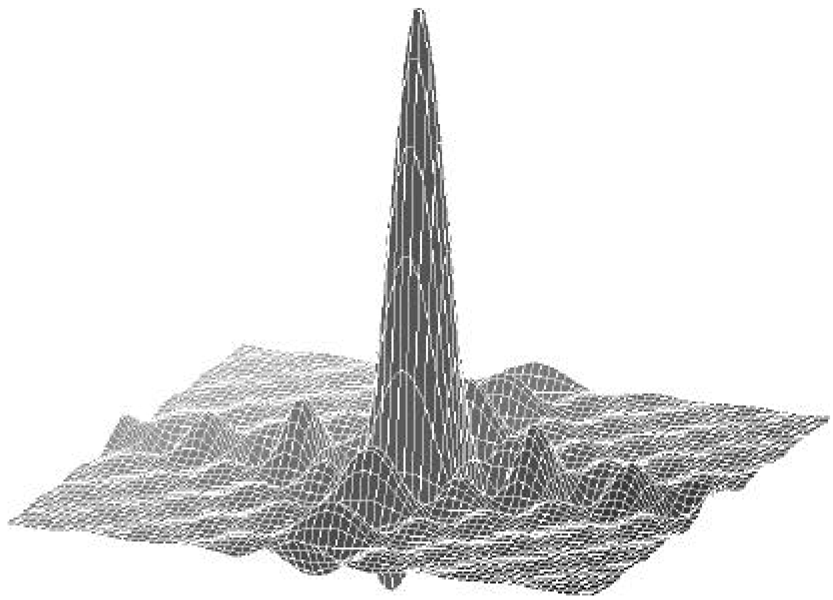
$$F(s_x,s_y) = \int_{-\infty}^{\infty} f_x(x)\exp(2pis_x x)dx \int_{-\infty}^{\infty} f_y(y)\exp(2pis_y y)dy$$

Example: What is the Fourier Transform of a 2-dimensional square tophat of unit width?



Solution: A square tophat can be written as a separable function. If a 1-d tophat function of unit width can be written as $\prod(x)$ then a 2-d square tophat function can be written as $f(x,y) = \prod(x)\prod(y)$. The Fourier Transform is then

$$F(s_x,s_y) = \text{sinc}(ps_x)\text{sinc}(ps_y)$$



3.13 Convolutions in higher dimensions

These are a natural extension of convolutions in 1-D. In 2-D we have

$$\begin{aligned} h(x,y) &= f(x,y) * g(x,y) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)g(x-u,y-v)\,du\,dv \end{aligned}$$

Writing this in vector form with $\vec{x} = (x, y)$ and $\vec{u} = (u, v)$ we have

$$h(\vec{x}) = \int_{\text{all space}} f(\vec{u})g(\vec{x}-\vec{u})\,dA$$

An example of the use of convolution in 3-D is to describe a crystal, with an arrangement of atoms repeated in space. If we write down a function for, say, the distribution of electrons in space, we can write it as the convolution of a function representing the distribution of electrons in a unit cell, and an array of 3-D delta functions which describe the layout of the unit cells

3.14 Deconvolution

Another application of the convolution theorem is in **deconvolution**.
In many physical systems, a signal can go through a "blurring" process of one sort or another which reduces the amount of detail which is discernible. Examples of this are:

- In 1-D: the transmission of radio or TV signals where the signal can take multiple different paths between the transmitter and the receiver due to reflections off buildings etc. What the receiver then sees is the sum of multiple copies of the signal with different time delays – "ghosting".

- In 2-D: imaging systems where imperfections in the optics etc blur the detail in the image.

We would like to be able to remove the effects of blurring by "undoing" the convolution. We can do this deconvolution easily in the Fourier domain: convolution in the Fourier domain is a multiplication, so deconvolution is just a division.

$$\begin{aligned} o(x) &= i(x) * r(x) \\ \Rightarrow O(s) &= R(s)I(s) \\ \therefore I(s) &= \frac{O(s)}{R(s)} \\ \Rightarrow i(x) &= \mathrm{I.F.T}\left\{\frac{\mathrm{F.T.}[\mathrm{o(x)}]}{\mathrm{F.T.}[\mathrm{r(x)}]}\right\} \end{aligned}$$

In practice, deconvolution is not as simple as this, because the wherever the frequency response function of the system is low, we end up multiplying the data by a large number, and this tends to amplify the noise in the image. In the case where the frequency response function goes to zero, we cannot do deconvolution at those frequencies: information has been lost.